Answer Chapter 2

Gamma and Beta Functions

Definitions:

We define the gamma function as

$$\Gamma(n) = \int_0^{\infty} e^{-t}\, t^{n-1}\, dt\,, \quad x > 0 \qquad (1)$$

and Beta function as

$$B(m,n) = \int_0^{1} t^{m-1}\,(1-t)^{n-1} dt, \qquad m > 0,\ n > 0 \qquad (2)$$

Properties of the Beta and Gamma functions:

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n\Gamma(n) \qquad \Gamma(n+1) = n!$$

$$\Gamma(n) = 2\int_0^{\infty} e^{-t^2}\, t^{2n-1}\, dt$$

$$\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta\, \sin^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\int_0^{\infty} e^{-t^2} dt = \frac{1}{2}\sqrt{\pi}$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad B(m,n) = B(n,m)$$

$$(i)\ B(m+1,n) = \frac{m}{m+n}\, B(m,n) \qquad (ii)\ B(m,n+1) = \frac{n}{m+n}\, B(m,n)$$

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\Gamma(n+\frac{1}{2}) \qquad \Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n}\, n!}\sqrt{\pi}$$

$\Gamma(m) = \infty$ if **m** is zero or a negative integer.

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin \pi n}$$

Exercises

(1) Prove that

(i) $\int_0^\infty e^{-ax} x^n \, dx = \frac{1}{a^{n+1}} \Gamma(n+1), \quad n > -1, a > 0$

Answer:

Let $t = ax$ then $dt = adx$

And $\int_0^\infty e^{-ax} x^n \, dx = \int_0^\infty e^{-t} \left(\frac{t}{a}\right)^n \frac{dt}{a} = \frac{1}{a^{n+1}} \int_0^\infty e^{-t} t^n \, dt = \frac{1}{a^{n+1}} \Gamma(n+1)$

(ii) $\int_0^\infty x^m e^{-x^n} \, dx = \frac{1}{n} \Gamma\{m + 1/n\}, \quad m > -1, n > 0$

Answer:

Let $t = x^n$ then $dt = nx^{n-1} dx$

and

$$\int_0^\infty x^m e^{-x^n} \, dx = \int_0^\infty t^{\frac{m}{n}} \frac{e^{-t}}{nt^{n-1/n}} dt = \frac{1}{n} \int_0^\infty t^{\frac{m+1-n}{n}} e^{-t} \, dt = \frac{1}{n} \Gamma\{m + 1/n\}$$

(iii) $\int_a^\infty \exp(2ax - x^2) dx = \frac{1}{2} \sqrt{\pi} \exp(a^2)$

Answer:

Let $(2ax - x^2) = a^2 - t$ then $2ax - x^2 - a^2 = -t$ and

$(x + a)^2 = t$

$2(x + a) dx = dt$

$2\sqrt{t} dx = dt$

$dx = \frac{1}{2\sqrt{t}} dt$

and

$$\int_a^\infty \exp(a^2 - t) \frac{1}{2\sqrt{t}} dt = \frac{1}{2} \exp(a^2) \int_a^\infty e^{-t} t^{-1/2} dt = \frac{1}{2} \exp(a^2) \Gamma(\frac{1}{2}) = \frac{1}{2} \sqrt{\pi} \exp(a^2)$$

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(2) Prove that

$$\int_0^{\pi/2} \tan^n \theta d\theta = \frac{1}{2}\Gamma\{(1+n)/2\}\Gamma\{(1-n)/2\} \text{ if } |n|<1$$

Answer:

$$\int_0^{\pi/2} \tan^n \theta d\theta = \int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^n \theta d\theta = \int_0^{\pi/2} \sin^n \theta \cos^{-n} \theta d\theta = \frac{1}{2}\Gamma\{(1+n)/2\}\Gamma\{(1-n)/2\}$$

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(3) Prove that

$$\int_0^{\pi/2} \sin^n \theta d\theta = \int_0^{\pi/2} \cos^n \theta d\theta = \frac{\sqrt{\pi}}{2}\frac{\Gamma\{(1+n)/2\}}{\Gamma\{(2+n)/2\}}$$

Answer:

$$\int_0^{\pi/2} \sin^n \theta d\theta = \int_0^{\pi/2} \sin^n \theta (\cos\theta)^0 d\theta = \frac{1}{2}B(\frac{n+1}{2},\frac{1}{2}) = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\{(1+n)/2\}}{\Gamma\{(2+n)/2\}}$$

$$\int_0^{\pi/2} \cos^n \theta d\theta = \int_0^{\pi/2} \cos^n \theta (\sin\theta)^0 d\theta = \frac{1}{2}B(\frac{1}{2},\frac{n+1}{2}) = \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+2}{2})} = \frac{\sqrt{\pi}}{2}\frac{\Gamma\{(1+n)/2\}}{\Gamma\{(2+n)/2\}}$$

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(4) Express each of the following integrals in terms of the Gamma or Beta functions and simplify when it possible:

(i) $\int_0^1 \left(\frac{1}{x}-1\right)^{1/4} dx$

Answer:

$$\int_0^1 \left(\frac{1}{x}-1\right)^{1/4} dx = \int_0^1 \left(\frac{1-x}{x}\right)^{1/4} dx = \int_0^1 (1-x)^{1/4} x^{-1/4} dx = B(\frac{5}{4},\frac{3}{4})$$

(ii) $\int_a^b (b-x)^{m-1}(x-a)^{n-1}dx, \ (b>a, m>0, n>0)$

Answer:

Let $y = \frac{x-a}{b-a}$ At $x=a$ we find $y=0$ and when $x=b$ we find $y=1$

$$dy=\frac{dx}{b-a}\quad\Rightarrow\quad dx=(b-a)dy$$

$x=y(b-a)+a$ Substitute in the integration

$$\int_a^b (b-x)^{m-1}(x-a)^{n-1}dx$$

$$=\int_0^1 \left[(b-(b-a)y-a\right]^{m-1}\left[(b-a)y+a-a\right]^{n-1}(b-a)dy$$

$$=\int_0^1 \left[(b-a)-(b-a)y\right]^{m-1}\left[(b-a)y\right]^{n-1}(b-a)dy$$

$$=\int_0^1 (b-a)^{m+n-2}\left(1-y\right)^{m-1}y^{n-1}dy=(b-a)^{m+n-2}\mathrm{B}(n,m)$$

(iii) $\int_0^1 x^m(1-x^n)^p dx,\quad (m>-1, p>-1, n>0)$

Answer:

Let $y=x^n$ then $x=y^{\frac{1}{n}}$ *and* $dx=\frac{1}{n}y^{\frac{1}{n}-1}dy$ Substitute in the integral then

$$\int_0^1 \frac{1}{n}y^{\frac{m}{n}}(1-y)^p\frac{1}{n}y^{\frac{1-n}{n}}dy=\frac{1}{n}\int_0^1 y^{\frac{m+n-1}{n}}(1-y)^p\frac{1}{n}dy=\frac{1}{n}\Gamma\left(\frac{m-1}{n},p+1\right)$$

(iv) $\int_0^1 \left(\ln\frac{1}{x}\right)^{a-1}dx,\quad (a>0)$

Answer:

Let $\frac{1}{x}=e^t$ then $x=e^{-t}$ *and* $dx=-e^{-t}dt$ Substitute in the integral then

$$\int_0^1 \left(\ln\frac{1}{x}\right)^{a-1}dx=-\int_\infty^0 (t)^{a-1}e^{-t}dx=\int_0^\infty (t)^{a-1}e^{-t}dx=\Gamma(a)$$

(v) $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}},\qquad (n>0)$

Answer:

Let $y=x^n$ then $x=y^{\frac{1}{n}}$ *and* $dx=\frac{1}{n}y^{\frac{1}{n}-1}dy$

Substitute in the integral then

$$\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \int_0^1 \frac{1}{n}\frac{dy}{\sqrt{(1-y)}}\ y^{\frac{1}{n}-1}dy = \frac{1}{n}\int_0^1 (1-y)^{-1/2}\ y^{\frac{1}{n}-1}dy = \frac{1}{n}\mathrm{B}(\frac{1}{2},\frac{1}{n})$$

(vi) $\displaystyle\int_0^\infty \frac{dt}{\sqrt{t}\ (1+t)}$

Answer:

Let $y = \dfrac{1}{1+t}$ when $t=0\ \ y=1$ and when $t=\infty\ \ y=0$

and $t = \left(\dfrac{1}{y}-1\right)$ *and* $dt = -\dfrac{1}{y^2}dy$

$t = \left(\dfrac{1-y}{y}\right)$ *and* $(1+t) = \dfrac{1}{y}$ Substitute in the integration then

$$\int_0^\infty \frac{dt}{\sqrt{t}\ (1+t)} = \int_0^\infty t^{-1/2}(1+t)^{-1}dt = -\int_1^0 \left(\frac{1-y}{y}\right)^{-1/2}\left(\frac{1}{y}\right)^{-1}\frac{1}{y^2}dy$$

$$= \int_0^1 (1-y)^{-1/2}\ y^{-1/2}dy = B(\frac{1}{2},\frac{1}{2})$$

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(5) Show that the area enclosed by the curve $x^4 + y^4 = 1$ is $\dfrac{\{\Gamma(\frac{1}{4})\}^2}{2\sqrt{\pi}}$.

Answer:

The required area is $\displaystyle\int_0^1 y dx = 4\int_0^1 (1-x^4)^{1/4}dx$

Let $t = x^4$ then $x = t^{1/4}$ and $dx = \dfrac{1}{4}t^{-3/4}dt$ when $x=0\ \ t=0$ and when $x=1\ \ t=1$

Substitute in the integration then

$$4\int_0^1 (1-x^4)^{1/4}dx = \int_0^1 (1-t)^{1/4}t^{-3/4}dt = B(\tfrac{5}{4},\tfrac{1}{4}) = \frac{\Gamma(\frac{5}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{6}{4})} = \frac{\frac{1}{4}\Gamma(\frac{1}{4})\Gamma(\frac{1}{4})}{\frac{2}{4}\Gamma(\frac{1}{2})} = \frac{1}{2}\frac{\left(\Gamma(\frac{1}{4})\right)^2}{\sqrt{\pi}}$$

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(6) Evaluate $\Gamma(\frac{-1}{2})$ and $\Gamma(\frac{-7}{2})$.

Answer:

Since $\Gamma(n+1) = n\Gamma(n)$ then $\Gamma(n) = \dfrac{1}{n}\Gamma(n+1)$

$$\Gamma(-\tfrac{1}{2}) = -2\Gamma(\tfrac{1}{2}) = -2\sqrt{\pi}$$

$$\Gamma(-\tfrac{7}{2}) = \frac{1}{(-\frac{7}{2})}\Gamma(-\tfrac{5}{2}) = \frac{1}{(-\frac{7}{2})}\frac{1}{(-\frac{5}{2})}\Gamma(-\tfrac{3}{2}) = \frac{1}{(-\frac{7}{2})}\frac{1}{(-\frac{5}{2})(-\frac{3}{2})}\Gamma(-\tfrac{1}{2})$$

$$= \frac{1}{(-\frac{7}{2})(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})}\Gamma(\tfrac{1}{2}) = \frac{16}{105}\sqrt{\pi}$$

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(7) Show that: (i) $\Gamma(x)\Gamma(-x) = \dfrac{-\pi}{x\sin\pi x}$

Answer:

Since $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin\pi x}$ and $\Gamma(x+1) = x\Gamma(x)$

then $\Gamma(1-x) = -x\Gamma(-x)$ and

$$\Gamma(x)\Gamma(1-x) = -x\Gamma(x)\Gamma(-x) = \frac{\pi}{\sin\pi x} \qquad \therefore \Gamma(x)\Gamma(-x) = \frac{-\pi}{x\sin\pi x}$$

(ii) $\Gamma(\tfrac{1}{2}-x)\Gamma(\tfrac{1}{2}+x) = \dfrac{\pi}{\cos\pi x}$

Answer:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x} \qquad (*)$$

put $x = \frac{1}{2} - n$ then $1 - x = 1 - (\frac{1}{2} - n) = \frac{1}{2} + n$ substitute in (*) we have

$$\Gamma(\tfrac{1}{2}-n)\Gamma(\tfrac{1}{2}+n) = \frac{\pi}{\sin\pi(\frac{1}{2}-n)} = \frac{\pi}{\sin(\frac{\pi}{2}-n\pi)} = \frac{\pi}{\sin\frac{\pi}{2}\cos n\pi - \cos\frac{\pi}{2}\sin n\pi} = \frac{\pi}{\cos n\pi}$$

or $\Gamma(\tfrac{1}{2}-x)\Gamma(\tfrac{1}{2}+x) = \dfrac{\pi}{\cos\pi x}$

Answers of Supplementary Problems

The Gamma function

(1)Evaluate

(a) $\dfrac{\Gamma(7)}{2\Gamma(4)\Gamma(3)} = \dfrac{6!}{2(3!)(2!)} = 30$

(b) $\dfrac{\Gamma(3)\Gamma(\frac{3}{2})}{\Gamma(\frac{9}{2})} = \dfrac{(2!)\,\Gamma(\frac{3}{2})}{(\frac{7}{2})(\frac{5}{2})(\frac{3}{2})\Gamma(\frac{3}{2})} = \dfrac{16}{105}$

(c) $\Gamma(1/2)\Gamma(\tfrac{3}{2})\Gamma(\tfrac{5}{2}) = \Gamma(\tfrac{1}{2})[(\tfrac{1}{2})\Gamma(\tfrac{1}{2})][(\tfrac{3}{2})(\tfrac{1}{2})\Gamma(\tfrac{1}{2})] = \dfrac{3}{8}[\Gamma(\tfrac{1}{2})]^3 = \dfrac{3}{8}\pi^{3/2}$

(2)Evaluate

(a) $\int_0^\infty x^4 e^{-x} dx = \Gamma(5) = 4! = 24$

(b) $\int_0^\infty x^6 e^{-3x} dx$ let $y = -3x$ we find $y: 0 \to \infty$

$\therefore \int_0^\infty x^6 e^{-3x} dx = \int_0^\infty (\frac{y}{3})^6 e^{-y} (\frac{dy}{3}) = \frac{1}{3^7} \int_0^\infty y^6 e^{-y} dy = \frac{\Gamma(7)}{3^7} = \frac{6!}{3^7} = \frac{80}{243}$

(c) $\int_0^\infty x^2 e^{-2x^2} dx$ let $y = 2x^2 \Rightarrow x = \frac{\sqrt{y}}{\sqrt{2}} \Rightarrow dx = \frac{dy}{2\sqrt{2}\sqrt{y}}$ *and* $y: 0 \to \infty$

$\therefore \int_0^\infty x^2 e^{-2x^2} dx = \int_0^\infty (\frac{y}{2}) e^{-y} (\frac{dy}{2\sqrt{2}\sqrt{y}}) = \frac{1}{4\sqrt{2}} \int_0^\infty y^{1/2} e^{-y} dy = \frac{\Gamma(1)}{4\sqrt{2}} = \frac{1}{4\sqrt{2}}$

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(3)Find (a) $\int_0^\infty e^{-x^3} dx$

Answer:

Let $y = x^3 \Rightarrow x = y^{1/3} \Rightarrow dx = \frac{1}{3} y^{-2/3} dy$, $y: 0 \to \infty$

$\therefore \int_0^\infty e^{-x^3} dx = \int_0^\infty e^{-y} (\frac{1}{3} y^{-2/3} dy) = \frac{1}{3} \int_0^\infty y^{-2/3} e^{-y} dy = \frac{1}{3} \Gamma(\frac{1}{3})$

(b) $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}} dx$

Answer:

Let $y = \sqrt{x} \Rightarrow x = y^2 \Rightarrow dx = 2y\, dy$, $y: 0 \to \infty$

$\therefore \int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}} dx = \int_0^\infty y^{1/2} e^{-y} (2y\, dy) = 2\int_0^\infty y^{3/2} e^{-y} dy = 2\Gamma(5/2)$

$= 2(3/2)(1/2)\Gamma(1/2) = \frac{3\sqrt{\pi}}{2}$

(c) $\int_0^\infty y^3 e^{-2y^5} dy$

Answer:

Let $t = 2y^5 \Rightarrow y = \frac{1}{\sqrt[5]{2}} t^{1/5} \Rightarrow dy = \frac{(1/5)}{\sqrt[5]{2}} t^{-4/5} dt\ ,\ t: 0 \to \infty$

$$\therefore \int_0^\infty y^3 e^{-2y^5} dy = \int_0^\infty (\frac{t^{1/5}}{\sqrt[5]{2}}) e^{-t} (\frac{1}{5\sqrt[5]{2}} t^{-4/5} dt) = \frac{1}{5(\sqrt[5]{2})^2} \int_0^\infty t^{-3/5} e^{-t}\, dt = \frac{1}{5(\sqrt[5]{2})^2} \Gamma(2/5)$$

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(4)Show that $\int_0^\infty \frac{e^{-st}}{\sqrt{t}} dt = \sqrt{\frac{\pi}{s}}$

Answer:

Let $y = st \Rightarrow t = \frac{y}{s} \Rightarrow dt = \frac{dy}{s},\ \ y: 0 \to \infty$

$$\therefore \int_0^\infty \frac{e^{-st}}{\sqrt{t}} dt = \int_0^\infty \sqrt{\frac{s}{y}}\, e^{-y} \frac{dy}{s} = \frac{1}{\sqrt{s}} \int_0^\infty y^{-1/2} e^{-y} dy = \frac{1}{\sqrt{s}} \Gamma(1/2) = \sqrt{\frac{\pi}{s}}$$

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(5)Prove that (a) $\int_0^1 (\ln\frac{1}{x})^{n-1} dx = \Gamma(n)$

Answer:

Let $y = \ln\frac{1}{x} = -\ln x \Rightarrow x = e^{-y} \Rightarrow dx = -e^{-y}\, dy\ ,\ \ y: \infty \to 0$

$$\therefore \int_0^1 (\ln\frac{1}{x})^{n-1} dx = \int_\infty^0 y^{n-1}(-e^{-y}\, dy) = \int_0^\infty y^{n-1} e^{-y}\, dy = \Gamma(n)$$

(b) $\int_0^1 x^p (\ln\frac{1}{x})^q dx = \frac{\Gamma(p+1)}{(p+1)^{q+1}}$

Answer:

Let $y = \ln\frac{1}{x} = -\ln x \Rightarrow x = e^{-y} \Rightarrow dx = -e^{-y}\, dy\ ,\ \ y: \infty \to 0$

$$\therefore \int_0^1 x^p (\ln\frac{1}{x})^q dx = \int_\infty^0 e^{-py} y^q (-e^{-y}\, dy) = \int_0^\infty y^q e^{-(p+1)y}\, dy$$

Let $t=(p+1)y \Rightarrow y=\frac{t}{p+1} \Rightarrow dy=\frac{dt}{p+1}$, $t:0\to\infty$

$$\Rightarrow \int_0^\infty y^q\, e^{-(p+1)y}\, dy=\int_0^\infty (\frac{t}{p+1})^q\, e^{-t}\, \frac{dt}{p+1}=\frac{1}{(p+1)^{q+1}}\int_0^\infty t^q\, e^{-t}\, dt=\frac{\Gamma(p+1)}{(p+1)^{q+1}}$$

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(6) Evaluate (a) $\int_0^1 (\ln x)^4 dx$

Answer:

Let $y=-\ln x \Rightarrow x=e^{-y} \Rightarrow dx=-e^{-y}\, dy$, $y:\infty\to 0$

$$\therefore \int_0^1 (\ln x)^4 dx=\int_\infty^0 (-y)^4\,(-e^{-y}\, dy)=\int_0^\infty y^4\, e^{-y}\, dy=\Gamma(5)=4!=24$$

(b) $\int_0^1 (x\,\ln x)^3 dx$

Answer:

Let $y=-\ln x \Rightarrow x=e^{-y} \Rightarrow dx=-e^{-y}\, dy$, $y:\infty\to 0$

$$\therefore \int_0^1 (x\,\ln x)^3 dx=\int_\infty^0 (-y\, e^{-y})^3\,(-e^{-y}\, dy)=-\int_0^\infty y^3\, e^{-4y}\, dy$$

Then put $t=4y \Rightarrow y=\frac{t}{4} \Rightarrow dy=\frac{dt}{4}$, $t:0\to\infty$

$$\Rightarrow -\int_0^\infty y^3\, e^{-4y}\, dy=-\int_0^\infty (\frac{t}{4})^3\, e^{-t}\, \frac{dt}{4}=-\frac{1}{4^4}\int_0^\infty t^3\, e^{-t}\, dt=-\frac{\Gamma(4)}{4^4}=-\frac{3!}{256}=-\frac{3}{128}$$

(c) $\int_0^1 \sqrt[3]{\ln(1/x)}\, dx$

Answer:

Let $y=\ln\frac{1}{x}=-\ln x \Rightarrow x=e^{-y} \Rightarrow dx=-e^{-y}\, dy$, $y:\infty\to 0$

$$\therefore \int_0^1 \sqrt[3]{\ln(1/x)}\, dx=\int_\infty^0 (-y)^{1/3}\,(-e^{-y}\, dy)=-\int_0^\infty y^{1/3} e^{-y}\, dy=-\Gamma(4/3)=-(1/3)\,\Gamma(1/3)$$

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(7) Evaluate

$(a)\ \Gamma(-7/2)$

$$\because\ \Gamma(n+1) = n\,\Gamma(n) \Rightarrow \Gamma(n) = \frac{1}{n}\Gamma(n+1)$$

$$\Gamma(-7/2) = \frac{\Gamma(-5/2)}{(-7/2)} = \frac{\Gamma(-3/2)}{(-7/2)(-5/2)} = \frac{\Gamma(-1/2)}{(-7/2)(-5/2)(-3/2)}$$

$$= \frac{\Gamma(1/2)}{(-7/2)(-5/2)(-3/2)(-1/2)} = \frac{16}{105}\sqrt{\pi}$$

$$(b)\ \ \Gamma(-1/3) = \frac{\Gamma(2/3)}{(-1/3)} = -3\,\Gamma(2/3)$$

Beta function

(8) Evaluate

$$\text{(a)}\ \ \mathrm{B}(3,5) = \frac{\Gamma(3)\Gamma(5)}{\Gamma(8)} = \frac{(2!)(4!)}{7!} = \frac{1}{105}$$

$$\text{(b)}\ \ \mathrm{B}(3/2,2) = \frac{\Gamma(3/2)\Gamma(2)}{\Gamma(7/2)} = \frac{\Gamma(3/2)\,(1!)}{(5/2)(3/2)\Gamma(3/2)} = \frac{4}{15}$$

$$\text{(c)}\ \ \mathrm{B}(1/3,2/3) = \frac{\Gamma(1/3)\Gamma(2/3)}{\Gamma(1)} = \frac{\pi}{\sin(\frac{\pi}{3})} = \frac{2\pi}{\sqrt{3}}$$

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(9) Find

$$\text{(a)}\ \ \int_0^1 x^2(1-x)^3 dx = \mathrm{B}(3,4) = \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{(2!)(3!)}{6!} = \frac{1}{60}$$

$$\text{(b)}\ \ \int_0^1 \sqrt{(1-x)/x}\,dx = \int_0^1 x^{-1/2}(1-x)^{1/2}dx = \mathrm{B}(1/2,3/2)$$

$$= \frac{\Gamma(1/2)\Gamma(3/2)}{\Gamma(2)} = \frac{\Gamma(1/2)\,[(1/2)\Gamma(1/2)]}{1!} = \frac{\pi}{2}$$

$$\text{(c)}\ \ \int_0^2 (4-x^2)^{3/2}dx$$

$$\text{Let}\ y = \frac{x^2}{4} \Rightarrow x = 2\sqrt{y} \Rightarrow dx = \frac{dy}{\sqrt{y}},\quad y: 0 \to 1$$

$$\Rightarrow \int_0^2 (4-x^2)^{3/2}dx = 8\int_0^2 (1-\frac{x^2}{4})^{3/2}dx = 8\int_0^1 (1-y)^{3/2}\frac{dy}{\sqrt{y}} = 8\int_0^1 y^{-1/2}(1-y)^{3/2}dy$$

$$= 8\,\mathrm{B}(1/2,5/2) = \frac{\Gamma(1/2)\Gamma(5/2)}{\Gamma(3)} = \frac{\Gamma(1/2)[(3/2)(1/2)\Gamma(1/2)]}{2!} = \frac{3\pi}{8}$$

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(10) Evaluate

(a) $(a)\int_0^4 u^{3/2}(4-u)^{5/2}du$

Answer:

Let $y = \frac{u}{4} \Rightarrow u = 4y \Rightarrow du = 4\,dy\;,\quad y: 0 \to 1$

$$\Rightarrow \int_0^4 u^{3/2}(4-u)^{5/2}du = 2^5\int_0^4 u^{3/2}(1-\frac{u}{4})^{5/2}du = 2^5\int_0^1 (4y)^{3/2}(1-y)^{5/2}(4\,dy)$$

$$= 2^{10}\int_0^1 y^{3/2}(1-y)^{5/2}\,dy = 2^{10}\,\mathrm{B}(5/2,7/2) = 2^{10}\,\mathrm{B}(5/2,7/2)$$

$$= 2^{10}\frac{\Gamma(5/2)\Gamma(7/2)}{\Gamma(6)} = \frac{2^{10}}{5!}\frac{3}{2}\frac{1}{2}\frac{5}{2}.\frac{3}{2}\frac{1}{2}\pi = 12\pi$$

$(b)\int_0^3 \frac{dx}{\sqrt{3x-x^2}}$

Answer:

Let $y = \frac{x}{3} \Rightarrow x = 3y \Rightarrow dx = 3\,dy\;,\quad y: 0 \to 1$

$$\Rightarrow \int_0^3 \frac{dx}{\sqrt{3x-x^2}} = \int_0^3 \frac{dx}{\sqrt{3x}\sqrt{1-(x/3)}} = \int_0^1 \frac{3dy}{\sqrt{9y}\sqrt{1-y}}$$

$$= \int_0^1 y^{-1/2}(1-y)^{-1/2} = \mathrm{B}(1/2,1/2) = \pi$$

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(11) Prove that $\int_0^a \frac{dy}{\sqrt{a^4-y^4}} = \frac{\{\Gamma(1/4)\}^2}{4a\sqrt{2\pi}}$

Answer:

Let $t=\dfrac{y^4}{a^4}\Rightarrow y=a\,t^{1/4}\Rightarrow dy=\dfrac{a}{4}t^{-3/4}\,dt\ ,\ \ t:0\to 1$

$$\Rightarrow\int_0^a\frac{dy}{\sqrt{a^4-y^4}}=\frac{1}{a^2}\int_0^a\frac{dy}{\sqrt{1-(y^4/a^4)}}=\frac{1}{a^2}\int_0^1\frac{(a/4)\,t^{-3/4}\,dt}{\sqrt{1-t}}$$

$$=\frac{1}{4a}\int_0^1 t^{-3/4}(1-t)^{-1/2}\,dt=\frac{1}{4a}\,\mathrm{B}(\tfrac14,\tfrac12)$$

$$=\frac{1}{4a}\frac{\Gamma(\frac14)\Gamma(\frac12)}{\Gamma(\frac34)}=\frac{\sqrt{\pi}}{4a}\frac{\Gamma(\frac14)}{\Gamma(\frac34)}=\frac{\sqrt{\pi}}{4a}\frac{\{\Gamma(\frac14)\}^2}{\Gamma(\frac14)\,\Gamma(\frac34)}$$

$$=\frac{\sqrt{\pi}}{4a}\frac{\{\Gamma(\frac14)\}^2}{\pi/\sin(\frac{\pi}{4})}=\frac{\{\Gamma(\frac14)\}^2}{4a\sqrt{2\pi}}$$

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(12) Evaluate

$$(a)\ \int_0^{\pi/2}\sin^4\theta\cos^4\theta\,d\theta=\frac12\mathrm{B}(\tfrac52,\tfrac52)=\frac12\frac{\{\Gamma(\frac52)\}^2}{\Gamma(5)}$$

$$(b)\ \int_0^{2\pi}\cos^6\theta\ d\theta=4\int_0^{\pi/2}\cos^6\theta\ d\theta=2\,\mathrm{B}(\tfrac72,\tfrac12)=\frac{2\,\Gamma(\frac72)\,\Gamma(\frac12)}{\Gamma(4)}$$

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(13) Show that $\displaystyle\int_0^\infty\frac{x^{p-1}}{1+x}dx=\Gamma(p)\Gamma(p-1)$

Answer:

Let $y=\dfrac{x}{1+x}$ when $x=0$ we find $y=0$ and when $x\to\infty$ we find $y\to\infty$

and $x=\dfrac{y}{1-y},\ dx=\dfrac{1}{(1-y)^2}dy$

The given integral becomes

$$\int_0^\infty y^{p-1}(1-y)^{-p}dy=\mathrm{B}(p,1-p)=\frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)}=\frac{\pi}{\sin p\pi}$$

(14) Prove that $\int_0^\infty \frac{\cos x\ dx}{x^p} = \frac{\pi}{2\Gamma(p)\cos(p\pi/2)} \quad 0<p<1$

Answer:

Since $\int_0^\infty t^{n-1}e^{-tx}dt = \frac{(n-1)!}{x^n} = \frac{\Gamma(n)}{x^n}$ Then $\frac{1}{x^n} = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-tx}dt$

Multiply both side by $\cos x$

$$\frac{\cos x}{x^n} = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-tx}\cos x dt$$

Then integrate with respect to x

$$\int_0^\infty \frac{\cos x}{x^n}dx = \frac{1}{\Gamma(n)}\int_0^\infty\int_0^\infty t^{n-1}e^{-tx}\cos x dx dt = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}\left(\int_0^\infty e^{-tx}\cos x dx\right)dt$$

$$= \frac{1}{\Gamma(n)}\int_0^\infty \frac{t^n}{(1+t^2)}dt = \left(\text{Put } u = \frac{t^2}{(1+t^2)}\right) = \frac{1}{2}\frac{1}{\Gamma(n)}\frac{\Gamma(n)\Gamma(n+1)}{\Gamma(n)}$$

$$\frac{\pi}{2\Gamma(n)\sin(n+1)\frac{\pi}{2}} = \frac{\pi}{2\Gamma(n)\cos(n\pi/2)}$$

Answer of quiz (3) in Mathematics
First year Electric department (power branch)

$$\int_0^\infty \frac{x^2}{1+x^4}dx$$

put $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta} \to dx = \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta d\theta$

when $x = 0$ we find $\theta = 0$

when $x = \infty$ we find $\theta = \frac{\pi}{2}$

$$\int_0^\infty \frac{x^2}{1+x^4}dx = \int_0^{\pi/2} \frac{\tan^2\theta}{(1+\tan^2\theta)}\frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta d\theta = \frac{1}{2}\int_0^{\pi/2}\sqrt{\tan\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta d\theta=\frac{1}{4}B(\frac{3}{4},\frac{1}{4})=\frac{1}{4}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)}=\frac{1}{4}.\frac{\pi}{\sin(\pi/4)}=\frac{\sqrt{2}\pi}{4}=\frac{\pi}{2\sqrt{2}}$$

we use the fact $\Gamma(n)\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$

Another solution

Put $y=\dfrac{x^4}{1+x^4}$ **when** $x=0$ we find $y=0$ **when** $x=\infty$ we find

$$y=\lim\frac{x^4}{1+x^4}=1$$

And $x^4=\dfrac{t}{1-t}\rightarrow x=\left(\dfrac{t}{1-t}\right)^{1/4}\rightarrow dx=\dfrac{1}{4}\left(\dfrac{t}{1-t}\right)^{-3/4}\left(\dfrac{1}{1-t}\right)^2 dt$

$$\int_0^\infty\frac{x^2}{1+x^4}dx=\int_0^\infty\frac{x^4.x^{-2}}{1+x^4}dx=\int_0^1 t\left(\frac{t}{1-t}\right)^{-1/2}\frac{1}{4}\left(\frac{t}{1-t}\right)^{-3/4}\left(\frac{1}{1-t}\right)^2 dt$$

$$=\frac{1}{4}\int_0^1 t^{-1/4}(1-t)^{-3/4}dt=\frac{1}{4}B(\tfrac{1}{4},\tfrac{3}{4})=\tfrac{1}{4}\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})}{\Gamma(1)}=\frac{1}{4}\frac{\pi}{\sin(\pi/4)}=\frac{\sqrt{2}\pi}{4}=\frac{\pi}{2\sqrt{2}}$$

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(2) $\displaystyle\int_0^2 x\sqrt[3]{8-x^3}\,dx$

Put $x^3=8y\rightarrow x=2y^{1/3}\rightarrow dx=\dfrac{2}{3}y^{-2/3}dy$

$$\int_0^2 x\sqrt[3]{8-x^3}\,dx=\int_0^1 2y^{1/3}\sqrt[3]{8-8y}\,\frac{2}{3}y^{-2/3}dy=\frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3}$$

$$=\frac{8}{3}B\left(\frac{2}{3},\frac{1}{3}\right)=\frac{8}{3}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})}{\Gamma(1)}=\frac{8}{3}\frac{\pi}{\sin(\pi/3)}=\frac{16\pi}{9\sqrt{3}}$$

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(3) $\displaystyle\int_0^{\pi/2}\sin^6\theta d\theta=\frac{1}{2}B(\tfrac{7}{2},\tfrac{1}{2})=\frac{1}{2}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{\Gamma(4)}=\frac{(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\left(\Gamma(\frac{1}{2})\right)^2}{3!}=\frac{5\pi}{32}$

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(4) $\displaystyle\int_0^{\pi/2}\tan^6\theta d\theta=\int_0^{\pi/2}\sin^6\theta\cos^{-6}\theta d\theta=\frac{1}{2}B(\frac{7}{2},\frac{-5}{2})$

$$=\frac{1}{2}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{-5}{2})}{\Gamma(1)}=\frac{1}{2}(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\Gamma(\frac{1}{2}).(\frac{-2}{5})(\frac{-2}{3})(\frac{-2}{1})\Gamma(\frac{1}{2})=\frac{1}{2}\Gamma(\frac{1}{2}).\Gamma(\frac{1}{2})=\frac{-\pi}{2}$$

We use the facts

$\Gamma(\frac{7}{2})=(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\Gamma(\frac{1}{2}).$

since $\Gamma(n+1)=n\,\Gamma(n)$ then $\Gamma(n)=\frac{1}{n}\Gamma(n+1)$

$\Gamma(-\frac{5}{2})=(-\frac{2}{5})\Gamma(-\frac{3}{2})=(-\frac{2}{5})(-\frac{2}{3})\Gamma(-\frac{1}{2})=(-\frac{2}{5})(-\frac{2}{3})(-\frac{2}{1})\Gamma(\frac{1}{2})$

Special Function

(1) Evaluate

$(a)\ \int_0^{\pi/2} \sin^6\theta d\theta$ $\quad (b)\ \int_0^{\pi/2} \cos^4\theta d\theta$ $\quad (c)\ \int_0^{\pi/2} \cos^4\theta\cos^5\theta d\theta$

Solution

(a) Since $\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$

Consider

$2m-1=6 \quad \rightarrow m=\frac{7}{2}$

$2n-1=0 \quad \rightarrow n=\frac{1}{2}$

then

$$(b)\ \int_0^{\pi/2} \sin^6\theta d\theta=\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}=\frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{7}{2}+\frac{1}{2})}=\frac{\Gamma(\frac{7}{2})\Gamma(\frac{1}{2})}{2\Gamma(4)}$$

$$=\frac{(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2(3!)}=\frac{5\pi}{32}$$

$$(c)\ (a)\ \int_0^{\pi/2} \cos^4\theta d\theta=\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{2\Gamma(3)}=\frac{(\frac{3}{2})(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2(2!)}=\frac{3\pi}{16}$$
